Opinion and consensus formation in the presence of polyadic interactions

Jan Mölter, *Technical University of Munich*
jointly with Anastasia Golovin, Pia Steinmeyer & Christian Kuehn
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SIAM Conference on Applications of Dynamical Systems – Minisymposium: Dynamical Systems on Networks and Fractals

Slides

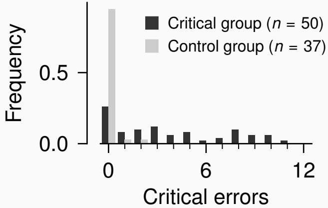
GROUP DYNAMICS IN OPINION FORMATION

Opinions are seldomly formed in isolation, but rather through **interactions** with others in society.

Besides pairwise interactions between individuals, **group dynamics** may play an integral part in opinion formation due to each individual's striving for conformity (bandwagon effect, peer-pressure ...).

Source: cottonbro studio

GROUP DYNAMICS IN OPINION FORMATION

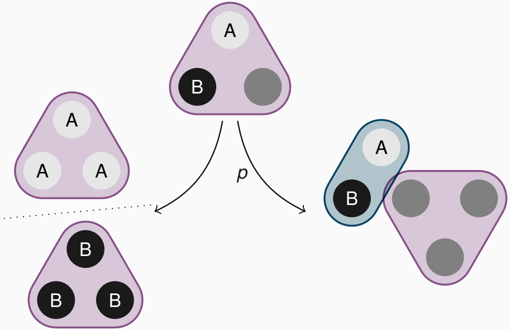


"The tendency to conformity in our society is so strong that reasonably intelligent and well-meaning young people are willing to call white black." *(Solomon Asch, 1955)*

Source: cottonbro studio

HYPERGRAPH ADAPTIVE VOTER MODEL

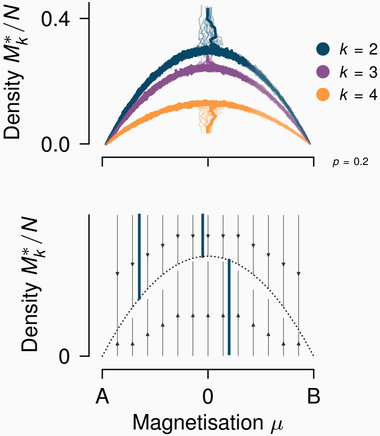
We generalise the (classical) adaptive voter model and use a **hypergraph** to encode polyadic interactions.



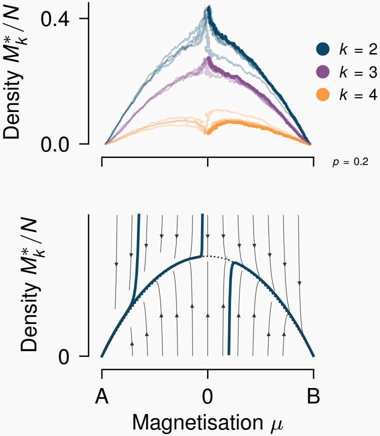
Adaptation: rewire-to-same or rewire-to-random.
Propagation: proportional voting or majority voting.

HYPERGRAPH ADAPTIVE VOTER MODEL: DYNAMICS

Proportional voting



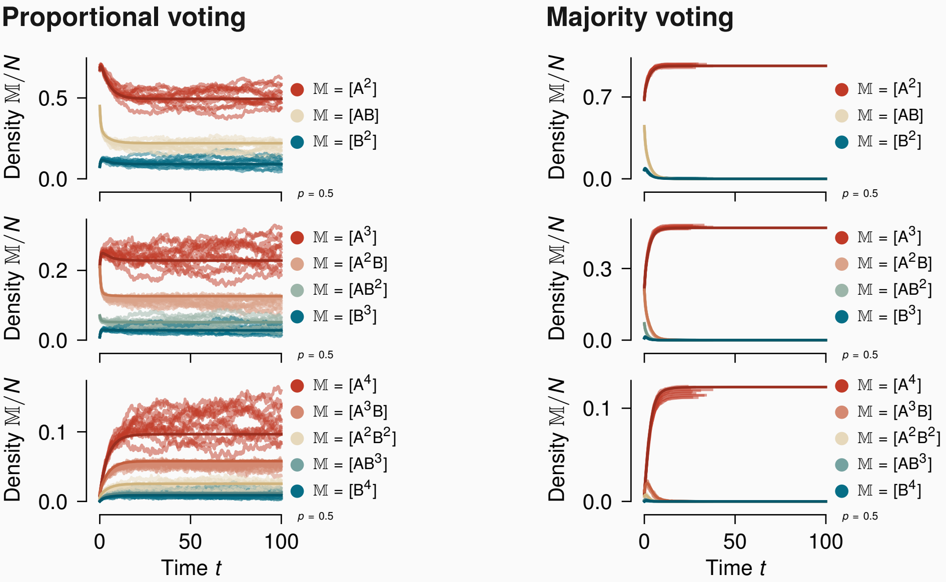
Majority voting



HYPERGRAPH ADAPTIVE VOTER MODEL: MEAN-FIELD DESCRIPTION FOR $K \geq 2$

$$
\begin{cases}
[\dot{\mathrm{A}}] = (1-p)\Big[\sum_{\substack{(\hat a,\hat b)\in\mathbb{N}^2\\ \hat a+\hat b\leq K}} [\mathrm{A}^{\hat a}\mathrm{B}^{\hat b}](\hat b\,\eta_{\mathrm{A}}(\hat a,\hat b) - \hat a\,\eta_{\mathrm{B}}(\hat a,\hat b))\Big] \\
[\dot{\mathrm{B}}] = (1-p)\Big[\sum_{\substack{(\hat a,\hat b)\in\mathbb{N}^2\\ \hat a+\hat b\leq K}} [\mathrm{A}^{\hat a}\mathrm{B}^{\hat b}](\hat a\,\eta_{\mathrm{B}}(\hat a,\hat b) - \hat b\,\eta_{\mathrm{A}}(\hat a,\hat b))\Big] \\
[\mathrm{A}^a\dot{}\mathrm{B}^b] = p\Big[\frac{a+1}{a+b+1}[\mathrm{A}^{a+1}\mathrm{B}^b]\,\mathbb{1}_{\mathbb{N}^2\cap\bar B^1_K(0)}(a+1,b) + \frac{b+1}{a+b+1}[\mathrm{A}^a\mathrm{B}^{b+1}]\,\mathbb{1}_{\mathbb{N}^2\cap\bar B^1_K(0)}(a,b+1) - [\mathrm{A}^a\mathrm{B}^b]\,\mathbb{1}_{\mathbb{N}^2\cap\bar B^1_K(0)}(a,b) \\
\qquad + \sum_{\substack{(\hat a,\hat b)\in\mathbb{N}^2\\ \hat a+\hat b\leq K}} [\mathrm{A}^{\hat a}\mathrm{B}^{\hat b}]\Big(\frac{\hat a}{\hat a+\hat b}(\pi_{\mathrm{A}}(a-1,b)-\pi_{\mathrm{A}}(a,b)) + \frac{\hat b}{\hat a+\hat b}(\pi_{\mathrm{B}}(a,b-1)-\pi_{\mathrm{B}}(a,b))\Big)\Big] \\
\qquad + (1-p)\Big[\sum_{\substack{(\hat a,\hat b)\in\mathbb{N}^2\\ \hat a+\hat b\leq K}} \eta_{\mathrm{A}}(\hat a,\hat b)\Big(\sum_{\substack{0\leq\alpha\leq\hat a\\ 1\leq\beta\leq\hat b\\ \cdots}} (1+\delta_{\hat a,a-\beta}\delta_{\hat b,b+\beta})[\mathrm{A}^{\hat a-\alpha}\mathrm{B}^{\hat b-\beta}(\mathrm{A}^\alpha\mathrm{B}^\beta)\mathrm{A}^{a-\alpha-\beta}\mathrm{B}^b] - \sum_{\substack{0\leq\alpha\leq\hat a\\ 1\leq\beta\leq\hat b\\ \cdots}} (1+\delta_{\hat a,a}\delta_{\hat b,b})[\mathrm{A}^{\hat a-\alpha}\mathrm{B}^{\hat b-\beta}(\mathrm{A}^\alpha\mathrm{B}^\beta)\mathrm{A}^{a-\alpha}\mathrm{B}^{b-\beta}]\Big) \\
\qquad + \sum_{\substack{(\hat a,\hat b)\in\mathbb{N}^2\\ \hat a+\hat b\leq K}} \eta_{\mathrm{B}}(\hat a,\hat b)\Big(\sum_{\substack{1\leq\alpha\leq\hat a\\ 0\leq\beta\leq\hat b\\ \cdots}} (1+\delta_{\hat a,a+\alpha}\delta_{\hat b,b-\alpha})[\mathrm{A}^{\hat a-\alpha}\mathrm{B}^{\hat b-\beta}(\mathrm{A}^\alpha\mathrm{B}^\beta)\mathrm{A}^a\mathrm{B}^{b-\alpha-\beta}] - \sum_{\substack{1\leq\alpha\leq\hat a\\ 0\leq\beta\leq\hat b\\ \cdots}} (1+\delta_{m,a}\delta_{\hat b,b})[\mathrm{A}^{\hat a-\alpha}\mathrm{B}^{\hat b-\beta}(\mathrm{A}^\alpha\mathrm{B}^\beta)\mathrm{A}^{a-\alpha}\mathrm{B}^{b-\beta}]\Big) \\
\qquad + \sum_{1\leq\alpha\leq a-1} \eta_{\mathrm{A}}(a-\alpha,\alpha)[\mathrm{A}^{a-\alpha}\mathrm{B}^\alpha]\,\delta_{b,0} + \sum_{1\leq\beta\leq b-1} \eta_{\mathrm{B}}(\beta,b-\beta)[\mathrm{A}^\beta\mathrm{B}^{b-\beta}]\,\delta_{a,0} - [\mathrm{A}^a\mathrm{B}^b]\,\mathbb{1}_{\mathbb{N}^2\cap\bar B^1_K(0)}(a,b)\Big] \quad \text{for } a,b\in\mathbb{N},\ a+b\leq K.
\end{cases}
$$

HYPERGRAPH ADAPTIVE VOTER MODEL: MEAN-FIELD TRAJECTORIES



HYPERGRAPH ADAPTIVE VOTER MODEL: MEAN-FIELD MAGNETISATION

$$\dot{\mu} = \frac{2(1-p)}{N} \sum_{\substack{(\hat{a},\hat{b})\in\mathbb{N}^2 \\ \hat{a}+\hat{b}\leq K}} [\mathrm{A}^{\hat{a}}\mathrm{B}^{\hat{b}}]\,(\hat{a}\,\eta_{\mathrm{B}}(\hat{a},\hat{b}) - \hat{b}\,\eta_{\mathrm{A}}(\hat{a},\hat{b}))$$

Proportional voting

Since $\eta_{\mathrm{A}}(\hat{a},\hat{b}) = \frac{\hat{a}}{\hat{a}+\hat{b}}$ and $\eta_{\mathrm{B}}(\hat{a},\hat{b}) = \frac{\hat{b}}{\hat{a}+\hat{b}}$,

$$\dot{\mu} = 0.$$

In equilibrium, there may be only **local consensus** (through fragmentation) or **no consensus** at all.

Majority voting

Since $\eta_{\mathrm{A}}(\hat{a},\hat{b}) = \Theta(\hat{a}-\hat{b})$ and $\eta_{\mathrm{B}}(\hat{a},\hat{b}) = \Theta(\hat{b}-\hat{a})$,

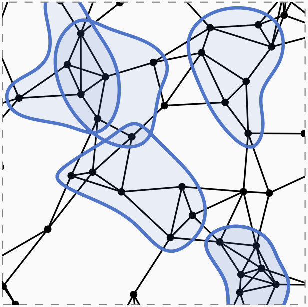
$$\dot{\mu} = \frac{2(1-p)}{N} \sum_{\substack{\sigma\in\mathbb{N}^2, |\sigma|\leq K \\ \sigma_1>\sigma_2}} \sigma_2\,([\mathrm{A}^{\sigma_2}\mathrm{B}^{\sigma_1}] - [\mathrm{A}^{\sigma_1}\mathrm{B}^{\sigma_2}]).$$

If $\mu \overset{(>)}{<} 0$, $[\mathrm{A}^{\sigma_2}\mathrm{B}^{\sigma_1}] - [\mathrm{A}^{\sigma_1}\mathrm{B}^{\sigma_2}] \overset{(>)}{<} 0$ and hence

$$\dot{\mu} \overset{(>)}{<} 0.$$

In equilibrium, there is always **global consensus**.

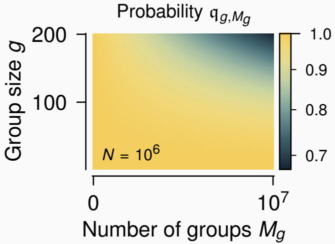
AD-HOC MODELLING OF GROUP DYNAMICS IN THE ADAPTIVE VOTER MODEL



We use a **hypergraph** to encode polyadic interactions in the sense that we consider an underlying network covered by M_g hyperegdes of size g.

A measure for the influence of the additional hypergraph structure is the probability that a randomly chosen edge is not contained in any of the hyperedges,

$$\mathfrak{q}_{g,M_g} = \left(1 - \frac{\binom{N-2}{g-2}}{\binom{N}{g}}\right)^{M_g} \approx e^{-\frac{M_g}{N^2} g(g-1)}.$$

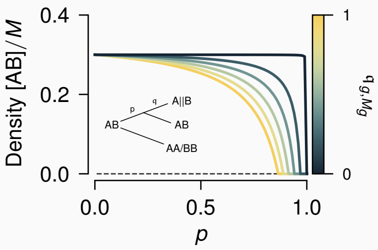


SCENARIO 1: STABILISING THE TOPOLOGY

Assumption: Rewiring only succeeds if the active egde is not contained in any of the groups given by the hyperedges.

$$\frac{1}{p_*} = 1 + \frac{1+\mu^2}{2(\langle k \rangle - 1)} \mathfrak{q}_{g,M_g}$$

Hence, $p_* \to 1$ exponentially fast as the hyperedges become larger and/or more.



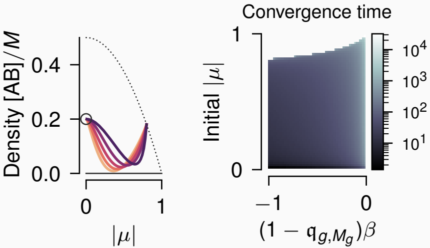
The inclusion in (social) groups prevents the topology from fragmentation, emphasising the importance of family or friendship groups for the functioning of a society.

SCENARIO 2: INTRODUCING BIAS TOWARDS MAJORITY OR MINORITY

Assumption: When contained in any of the groups given by the hyperedges, propagation occurs with a bias towards promoting majority or minority opinion.

$$\dot{\mu} = (1 - p)\frac{2(1 - \mathfrak{q}_{g,M_g})\beta}{N}\mu\,[\mathrm{AB}]$$

Hence, in any non-trivial equilibrium, necessarily $\mu = 0$ and $\beta \leq 0$.



Even the slightest institutional (majority) mistrust manifesting itself in a minority bias promoted by (social) groups leads to a functioning yet deeply divided society with no (stable) majority.

CONCLUSIONS

→ We have argued that it may be crucial to take **group dynamics** into account when studying opinion formation.

→ We have shown how to define a **proper generalisation** of the classical adaptive voter model for polyadic interactions, presented a mean-field description, and discussed how different dynamics may affect **consensus formation**.

→ We have discussed two **low-dimensional ad-hoc models** where polyadic interactions lead to interesting societal effects.

A. Golovin, J. Mölter, and C. Kuehn (2024). “Polyadic Opinion Formation: The Adaptive Voter Model on a Hypergraph”. **Ann. Phys. (Berlin) 536**:2300342. DOI: 10.1002/andp.202300342. arXiv: 2308.03640 [physics.soc-ph]

P. Steinmeyer, J. Mölter, and C. Kuehn (2025). “Phase transitions for polyadic epidemic and voter models with multiscale groups”. arXiv: 2410.12682 [nlin.AO]

Thank you!

Slides

Deutsche Forschungsgemeinschaft

VolkswagenStiftung

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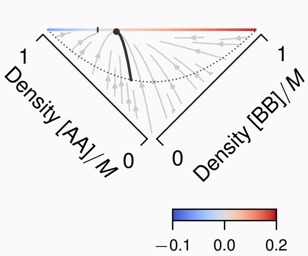
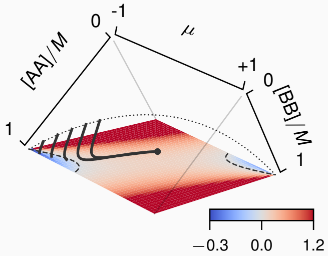
BISTABILITY IN THE ADAPTIVE VOTER MODEL





In the phase space given by the simplicial cylinder

$$[-1,+1] \times \{x \in \mathbb{R}^2 : x \geq 0 \wedge \|x\|_1 \leq M\},$$

the trivial equilibria form the manifold

$$[-1,+1] \times \{x \in \mathbb{R}^2 : x \geq 0 \wedge \|x\|_1 = M\}.$$

A trivial equilibrium $(\mu, (1-\theta)M, \theta M)$ is (linearly) stable if

$$-\frac{(1-\hat{\beta})\mu(\mu-(2\theta-1))}{1-\mu^2} > 1 - \frac{1-\frac{1}{2}p}{1-p}\frac{1}{\langle k \rangle}.$$