Preserving Bifurcations through Moment Closures

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Slides

MOMENT SYSTEMS & MOMENT CLOSURES

Moment systems are generically given as infinite-dimensional systems of the form

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, \dots x_\kappa, x_{\kappa+1}, \dots)\\ \dot{x}_2 &= f_2(x_1, x_2, \dots x_\kappa, x_{\kappa+1}, \dots)\\ \vdots &= \qquad \vdots\end{aligned}$$

A (moment) closure relation for some $\kappa \in \mathbb{N}$ is a mapping H such that

$$H(x_1, x_2, \dots x_\kappa) = (x_{\kappa+1}, x_{\kappa+2}, \dots).$$

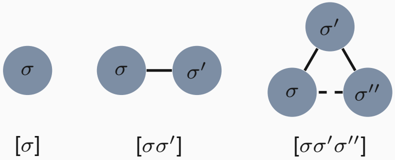
Through applying the closure relation, the original system is rendered the closed, finite-dimensional system

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, \dots x_\kappa, H(x_1, x_2, \dots x_\kappa))\\ \dot{x}_2 &= f_2(x_1, x_2, \dots x_\kappa, H(x_1, x_2, \dots x_\kappa))\\ \vdots &= \qquad \vdots\\ \dot{x}_\kappa &= f_\kappa(x_1, x_2, \dots x_\kappa, H(x_1, x_2, \dots x_\kappa))\end{aligned}$$

MOMENT SYSTEMS IN NETWORK DYNAMICAL SYSTEMS

Network dynamical systems frequently admit a mean-field description of *network moments*.

These correspond to the expected number of certain motifs.



Assuming binary-state dynamics, up to order 2, these mean-field moment systems are generically given as

$$\begin{cases} [\dot{\sigma_1}] = f_{[\sigma_1]}([\sigma_1], [\sigma_2], [\sigma_1\sigma_1], [\sigma_1\sigma_2], [\sigma_2\sigma_2], \ldots, \lambda) \\ [\dot{\sigma_2}] = f_{[\sigma_2]}([\sigma_1], [\sigma_2], [\sigma_1\sigma_1], [\sigma_1\sigma_2], [\sigma_2\sigma_2], \ldots, \lambda) \\ [\sigma_1\dot{\sigma}_1] = f_{[\sigma_1\sigma_1]}([\sigma_1], [\sigma_2], [\sigma_1\sigma_1], [\sigma_1\sigma_2], [\sigma_2\sigma_2], \ldots, \lambda) \\ [\sigma_1\dot{\sigma}_2] = f_{[\sigma_1\sigma_2]}([\sigma_1], [\sigma_2], [\sigma_1\sigma_1], [\sigma_1\sigma_2], [\sigma_2\sigma_2], \ldots, \lambda) \\ [\sigma_2\dot{\sigma}_2] = f_{[\sigma_2\sigma_2]}([\sigma_1], [\sigma_2], [\sigma_1\sigma_1], [\sigma_1\sigma_2], [\sigma_2\sigma_2], \ldots, \lambda) \end{cases}$$

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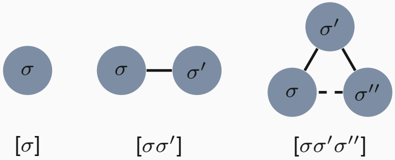

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\dot{[\sigma_1\sigma_1]} = f_{[\sigma_1\sigma_1]}([\sigma_1], [\sigma_2], [\sigma_1\sigma_1], [\sigma_1\sigma_2], [\sigma_2\sigma_2], \dots, \lambda) \\
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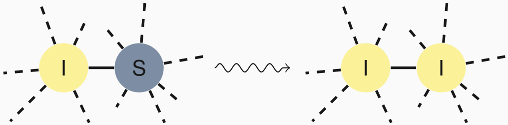
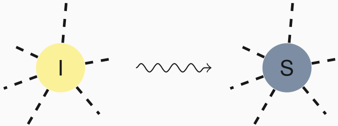
$$

THE SIS EPIDEMIC MODEL

The SIS epidemic is model for the spreading of a contagion without immunity.

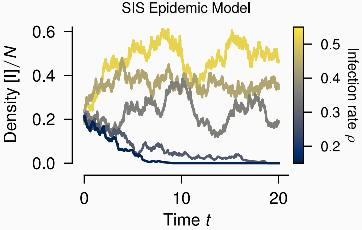


Infection



Recovery

$$\begin{cases} [\dot{S}] = [I] - \rho\,[SI] \\ [\dot{I}] = \rho\,[SI] - [I] \\ [\dot{SS}] = 2[SI] - 2\rho[SSI] \\ [\dot{SI}] = [II] - [SI] + \rho([SSI] - [ISI] - [SI]) \\ [\dot{II}] = -2[II] + 2\rho([ISI] + [SI]) \end{cases}$$



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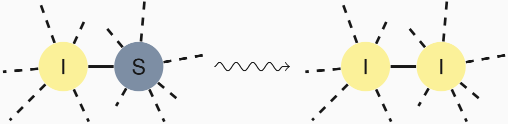

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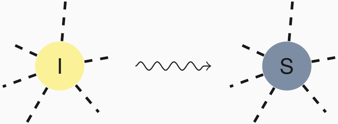

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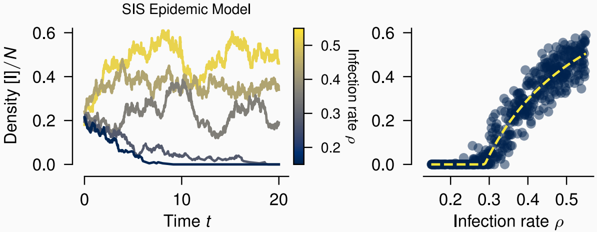

We consider the system

$$\begin{cases} [\dot{\mathrm{I}}] = \rho[\mathrm{SI}] - [\mathrm{I}] \\ [\dot{\mathrm{SI}}] = [\mathrm{II}] - [\mathrm{SI}] + \rho([\mathrm{SSI}] - [\mathrm{ISI}] - [\mathrm{SI}]) \\ [\dot{\mathrm{II}}] = -2[\mathrm{II}] + 2\rho([\mathrm{ISI}] + [\mathrm{SI}]) \end{cases}$$

subject to $[\mathrm{S}]+[\mathrm{I}] = N$ and $[\mathrm{SS}]+2[\mathrm{SI}]+[\mathrm{II}] = 2M$

and assume a closure relation H such that

$$([\mathrm{SSI}], [\mathrm{ISI}]) = H([\mathrm{I}], [\mathrm{SI}], [\mathrm{II}]).$$

Theorem

Assume that H is rational and that $H([\mathrm{I}], [\mathrm{SI}], [\mathrm{II}]) = 0$ whenever $[\mathrm{SI}] = 0$. Then H can be factorised so that $H([\mathrm{I}], [\mathrm{SI}], [\mathrm{II}]) = [\mathrm{SI}]\,\tilde{H}([\mathrm{I}], [\mathrm{SI}], [\mathrm{II}])$. Moreover, suppose that $\tilde{H}$ is at least twice continuously differentiable in a neighbourhood around 0 and that $\frac{1}{\rho_*} = \tilde{H}^{([\mathrm{SSI}])}(0) > 0$. Then, if

$$\partial_{[\mathrm{I}]}\tilde{H}^{([\mathrm{SSI}])}(0) + \tilde{H}^{([\mathrm{SSI}])}(0)\partial_{[\mathrm{SI}]}\tilde{H}^{([\mathrm{SSI}])}(0) + (1 + \tilde{H}^{([\mathrm{ISI}])}(0))\partial_{[\mathrm{II}]}\tilde{H}^{([\mathrm{SSI}])}(0) \neq 0,$$

the closed systems exhibits a transcritical bifurcation at $\rho = \rho_*$.

In particular, provided that $2\tilde{H}^{([\mathrm{SSI}])}(0) + \tilde{H}^{([\mathrm{ISI}])}(0) + 1 > 0$, the bifurcation is supercritical (subcritical) if

$$\partial_{[\mathrm{I}]}\tilde{H}^{([\mathrm{SSI}])}(0) + \tilde{H}^{([\mathrm{SSI}])}(0)\partial_{[\mathrm{SI}]}\tilde{H}^{([\mathrm{SSI}])}(0) + (1 + \tilde{H}^{([\mathrm{ISI}])}(0))\partial_{[\mathrm{II}]}\tilde{H}^{([\mathrm{SSI}])}(0) \overset{(>)}{<} 0.$$

Proof: Application of the Crandall–Rabinowitz Theorem. □

MOMENT CLOSURES PRESERVING THE BIFURCATION IN THE SIS EPIDEMIC MODEL

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$$\begin{cases} [\dot{\mathrm{I}}] = \rho[\mathrm{SI}] - [\mathrm{I}] \\ [\dot{\mathrm{SI}}] = [\mathrm{II}] - [\mathrm{SI}] + \rho([\mathrm{SSI}] - [\mathrm{ISI}] - [\mathrm{SI}]) \\ [\dot{\mathrm{II}}] = -2[\mathrm{II}] + 2\rho([\mathrm{ISI}] + [\mathrm{SI}]) \end{cases}$$

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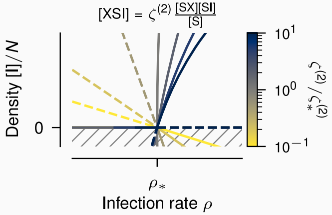
EXAMPLES & VALIDATION OF EXISTING CLOSURES

In case of the most frequently used closure

$$[XSI] = \zeta^{(2)} \frac{[XS][SI]}{[S]},$$

one obtains a transcritical bifurcation at $\rho_* = \frac{N}{2\zeta^{(2)}M}$ that is supercritical if $\zeta^{(2)} > \zeta_*^{(2)} := \frac{1}{2}(1 - \frac{N}{2M})$ and subcritical otherwise.



A similar result can be obtained for the closure ($\phi \neq 1$)

$$[XSI] = \zeta^{(2)} \frac{[XS][SI]}{[S]} \left(1 - \phi\left(1 - \zeta^{(1)} N \frac{[S][I][XI]}{([SS][I] + [S][II])[X][SI]}\right)\right).$$

Other closures such as $[XSI] = \xi[X][SI]$, $[XSI] = 0$, or $[XSI] = \xi([X] + [II])[SI]$ yield as transcritical bifurcation, no bifurcation, or a different bifurcation.

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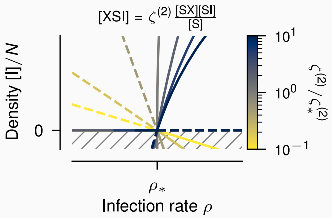

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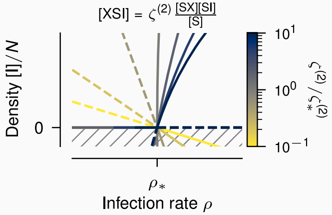

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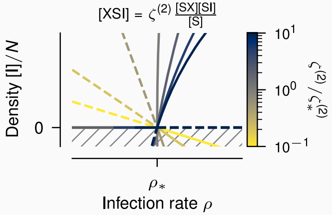

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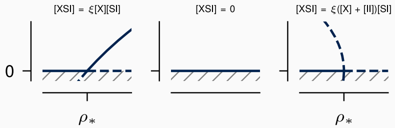

CONCLUSIONS

→ An alternative approach to find moment closures is to focus on qualitative features that are to be preserved locally first.

→ We demonstrated this in the context of a paradigmatic network dynamical systems and derived rigorous conditions on a moment closure to produe the expected bifurcation.

→ This classification of “good” moment closures in a principled way provides rigorous and quantitative evidence for the validity of existing moment closures.

Future work

→ Instead of only a single qualitative feature, we may combine several and this way further constrain “good” moment closures.

C. Kuehn and J. Mölter (2023). “Preserving Bifurcations through Moment Closures”. arXiv: 2307.03468 [nlin.AO]

Slides

Thank you!

TUM

DFG Deutsche Forschungsgemeinschaft

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