The Influence of a Transport Process on the Epidemic Threshold

Jan Mölter, *Technical University of Munich*; joint work with Christian Kuehn
12^{th} July 2022

SIAM Conference on the Life Sciences – Minisymposium: Dynamical Transitions in Nodes and Networks

TRANSPORT PROCESSES ACCELERATE EPIDEMIC DYNAMICS

Source: Jeffrey Young | zydeaøsika

Public transport transiently brings together people in a **confined space** and as such provides a genuine risk to spread contagions.

When modelling an epidemic, one classically considers the epidemic dynamics on a **static** social network. In contrast, transport dynamically generates **transient** connection between people.

→ How can we incorporate transport into the classical epidemic network models and quantify its effect e.g. on the epidemic threshold?

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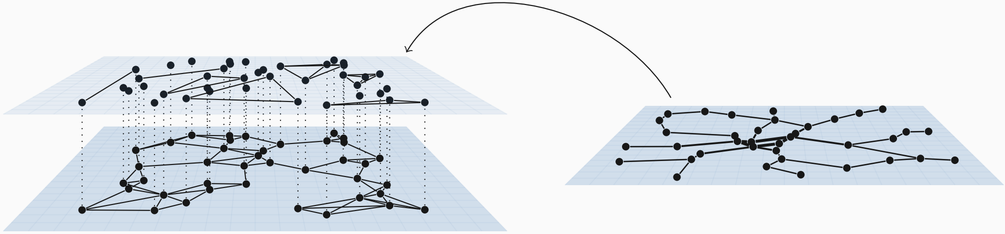
$\rightarrow$ How can we incorporate transport into the classical epidemic network models and quantify its effect e.g. on the epidemic threshold?

AN EPIDEMIC NETWORK MODEL INCORPORATING TRANSPORT



Epidemic (multilayer) network $(\mathcal{N}, \{\mathcal{E}^{\text{c.}}, \mathcal{E}^{\text{t.}}\})$

Transport network $(\mathcal{X}, \mathcal{A})$

Multiplex structure of simple networks, with a static bottom ("community") and a dynamic top ("transport") layer

$\rightarrow$ standard epidemic dynamics (**SIS**, SIR, …) across the entire multiplex

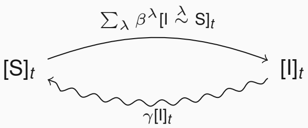
Simple, static network

$\rightarrow$ independent Poissonian random walks of the individuals from the population $\mathcal{N}$

$\rightarrow$ the event that two individuals occupy the same site generates a link between them in $\mathcal{E}^{\text{t.}}$

MEAN-FIELD DESCRIPTION FOR SIS-EPIDEMIC DYNAMICS UP TO 2ND ORDER

1st-order transition diagram



2nd-order transition diagram

- $\rightarrow$ β^ω infection rate in layer ω
- $\rightarrow$ γ recovery rate

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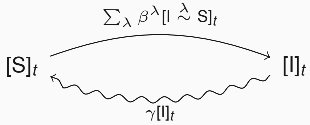

→ β^ω infection rate in layer ω

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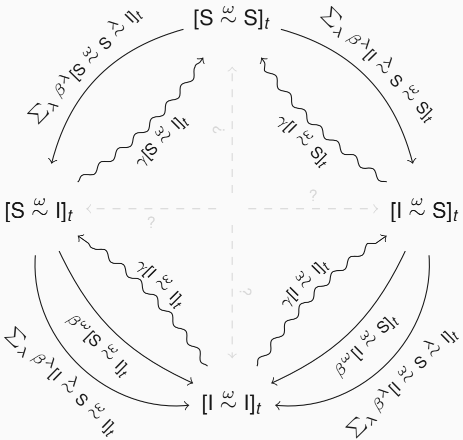

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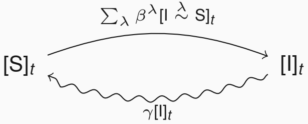
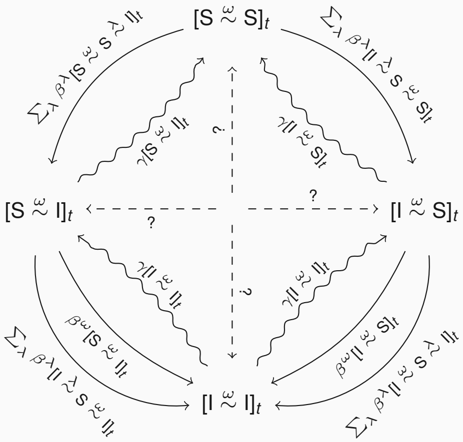


→ β^ω infection rate in layer ω

→ γ recovery rate

2^{nd}-order transition diagram



PAIR-MOTIF DYNAMICS DUE TO TRANSPORT — MICROSCOPIC LEVEL

Let H_t^n and X_t^n denote the health and location of individual n at time t. If $h_t(x)$ is the number of individuals in state h occupying site x at time t, then, for $\tau > 0$ sufficiently small,

$$h_{t+\tau}(x) = h_t(x) + \sum_n \left(\delta_{x,X_{t+\tau}^n} - \delta_{x,X_t^n}\right)\delta_{h,H_t^n}.$$

With that, if $\{h \overset{t.}{\sim} h'\}_t(x)$ is the number of links between individuals in state h and h' occupying site x at time t,

$$\{h \overset{t.}{\sim} h'\}_t(x) = h_t(x)\left(h'_t(x) - \delta_{h,h'}\right)$$

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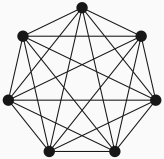

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The generator of the random walk on the transport network is the Laplacian $\Delta = \mathbb{1} - \mathrm{P}$ so that

$$\mathbb{P}\left[X^n_{t+\tau} = x' \,\middle|\, X^n_t = x \wedge H^n_t = h\right] = \delta_{x,x'} - \mu\tau\Delta(x,x')\left(1 + \mathcal{O}(\tau)\right).$$

Hence, in expectation and the limit $\tau \to 0$,

$$\partial_t\, [h(x)]_t = -\mu \sum_{x'} \Delta^\top(x,x')[h(x')]_t$$

and

$$\partial_t\, [\{h \overset{t.}{\sim} h'\}(x)]_t = -\mu\left([h'(x)]_t \sum_{x'} \Delta^\top(x,x')[h(x')]_t + [h(x)]_t \sum_{x'} \Delta^\top(x,x')[h'(x')]_t\right).$$

Finally, across all the sites of the network

$$\partial_t\, [h \overset{t.}{\sim} h']_t = -\mu \sum_{x,x'} \left(\Delta^\top(x,x') + \Delta^\top(x',x)\right)[h(x)]_t[h'(x')]_t.$$

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But, this equation depends on the number of individuals in a certain state occupying a specific site x!

Hence, assuming $[h(x)]_t \approx p_t(x)[h]_t$, instead

$$\begin{cases} \partial_t \, p_t = -\mu\Delta^\top p_t \\ \partial_t \, [h \overset{t.}{\sim} h']_t \approx \partial_t \, \|p_t\|^2 \, [h]_t[h']_t \end{cases}$$

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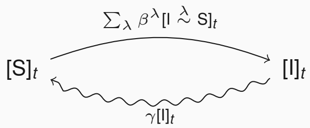
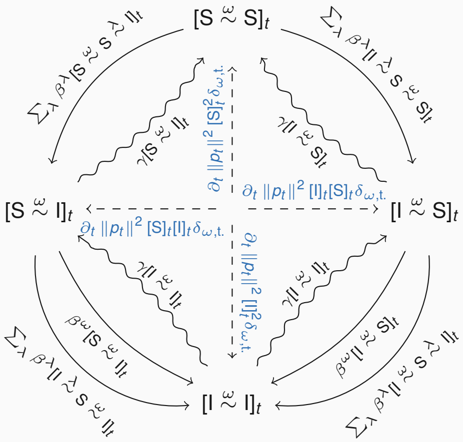


2nd-order transition diagram



MEAN-FIELD DESCRIPTION FOR SIS-EPIDEMIC DYNAMICS UP TO 2ND ORDER

$$
\begin{cases}
\partial_t\, p_t = -\mu \Delta^\top p_t \\
\partial_t\, [S]_t = -\sum_\lambda \beta^\lambda [S \overset{\lambda}{\sim} I]_t + \gamma [I]_t \\
\partial_t\, [I]_t = \sum_\lambda \beta^\lambda [S \overset{\lambda}{\sim} I]_t - \gamma [I]_t \\
\partial_t\, [S \overset{\omega}{\sim} S]_t = -2 \sum_\lambda \beta^\lambda [S \overset{\omega}{\sim} S \overset{\lambda}{\sim} I]_t + 2\gamma [S \overset{\omega}{\sim} I]_t + \partial_t \|p_t\|^2\, [S]_t^2 \delta_{\omega,\mathrm{t.}} \\
\partial_t\, [S \overset{\omega}{\sim} I]_t = \sum_\lambda \beta^\lambda \left([S \overset{\omega}{\sim} S \overset{\lambda}{\sim} I]_t - [I \overset{\omega}{\sim} S \overset{\lambda}{\sim} I]_t \right) - \beta^\omega [S \overset{\omega}{\sim} I]_t \\
\qquad - \gamma \left([S \overset{\omega}{\sim} I]_t - [I \overset{\omega}{\sim} I]_t \right) + \partial_t \|p_t\|^2\, [S]_t [I]_t \delta_{\omega,\mathrm{t.}} \\
\partial_t\, [I \overset{\omega}{\sim} I]_t = 2 \left(\sum_\lambda \beta^\lambda [I \overset{\omega}{\sim} S \overset{\lambda}{\sim} I]_t + \beta^\omega [S \overset{\omega}{\sim} I]_t \right) - 2\gamma [I \overset{\omega}{\sim} I]_t \\
\qquad + \partial_t \|p_t\|^2\, [I]_t^2 \delta_{\omega,\mathrm{t.}}
\end{cases}
$$

Assume the community layer of the epidemic network is k-regular and let

$$
\kappa_t^\omega = \begin{cases} k & \text{if } \omega = \mathrm{c.} \\ \|p_t\|^2\, |\mathcal{N}| & \text{if } \omega = \mathrm{t.} \end{cases}
$$

For a closure

- at the level of pairs, use

$$
[S \overset{\omega}{\sim} I]_t \approx \frac{\kappa_t^\omega}{|\mathcal{N}|} [S]_t [I]_t
$$

  and

- at the level of triples, use

$$
[h \overset{\omega}{\sim} S \overset{\omega'}{\sim} I]_t \approx \left(1 - \frac{\delta_{\omega,\omega'}}{\kappa_t^\omega} \right) \frac{[S \overset{\omega}{\sim} h]_t [S \overset{\omega'}{\sim} I]_t}{[S]_t}
$$

MEAN-FIELD DESCRIPTION FOR SIS-EPIDEMIC DYNAMICS UP TO 2^{ND} ORDER

$$
\begin{cases}
\partial_t \; p_t = -\mu \Delta^\top p_t \\
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\partial_t \; [\mathrm{S} \overset{\omega}{\sim} \mathrm{I}]_t = \sum_\lambda \beta^\lambda \left([\mathrm{S} \overset{\omega}{\sim} \mathrm{S} \overset{\lambda}{\sim} \mathrm{I}]_t - [\mathrm{I} \overset{\omega}{\sim} \mathrm{S} \overset{\lambda}{\sim} \mathrm{I}]_t \right) - \beta^\omega [\mathrm{S} \overset{\omega}{\sim} \mathrm{I}]_t \\
\qquad - \gamma \left([\mathrm{S} \overset{\omega}{\sim} \mathrm{I}]_t - [\mathrm{I} \overset{\omega}{\sim} \mathrm{I}]_t \right) + \partial_t \left\| p_t \right\|^2 [\mathrm{S}]_t [\mathrm{I}]_t \delta_{\omega,\mathrm{t.}} \\
\partial_t \; [\mathrm{I} \overset{\omega}{\sim} \mathrm{I}]_t = 2 \left(\sum_\lambda \beta^\lambda [\mathrm{I} \overset{\omega}{\sim} \mathrm{S} \overset{\lambda}{\sim} \mathrm{I}]_t + \beta^\omega [\mathrm{S} \overset{\omega}{\sim} \mathrm{I}]_t \right) - 2\gamma [\mathrm{I} \overset{\omega}{\sim} \mathrm{I}]_t \\
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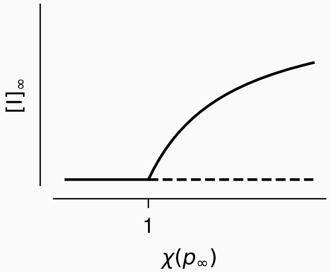
- at the level of triples, use

$$
[h \overset{\omega}{\sim} S \overset{\omega'}{\sim} \mathrm{I}]_t \approx \left(1 - \frac{\delta_{\omega,\omega'}}{\kappa_t^\omega} \right) \frac{[\mathrm{S} \overset{\omega}{\sim} h]_t [\mathrm{S} \overset{\omega'}{\sim} \mathrm{I}]_t}{[\mathrm{S}]_t}
$$

1^{ST}-ORDER MEAN-FIELD DESCRIPTION FOR SIS-EPIDEMIC DYNAMICS

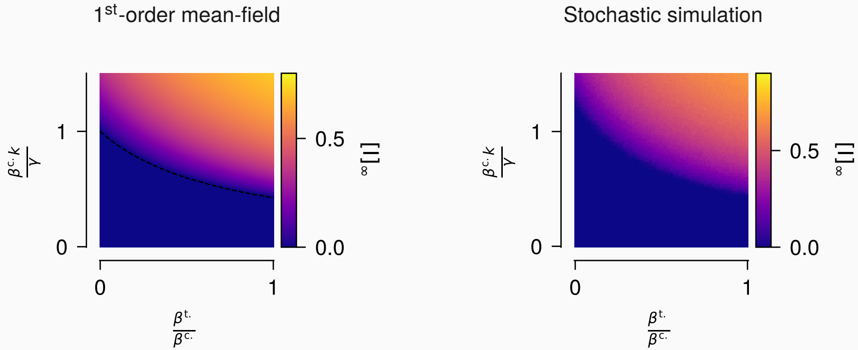
$$\begin{cases} \partial_t \, p_t = -\mu \Delta^\top p_t \\ \partial_t \, [\mathrm{S}]_t = -\left(\beta^{\text{c.}} \frac{k}{|\mathcal{N}|} + \beta^{\text{t.}} \, \|p_t\|^2\right) [\mathrm{S}]_t [\mathrm{I}]_t + \gamma [\mathrm{I}]_t \\ \partial_t \, [\mathrm{I}]_t = \left(\beta^{\text{c.}} \frac{k}{|\mathcal{N}|} + \beta^{\text{t.}} \, \|p_t\|^2\right) [\mathrm{S}]_t [\mathrm{I}]_t - \gamma [\mathrm{I}]_t \end{cases}$$

The system undergoes a **transcritical bifurcation** when $\chi(p_\infty) = 1$. Here, $\chi(p) = \left(\beta^{\text{c.}} \frac{k}{|\mathcal{N}|} + \beta^{\text{t.}} \, \|p\|^2\right) \frac{|\mathcal{N}|}{\gamma}$ and p_∞ is the unique probability distribution solving the equation $\Delta^\top p_\infty = 0$.



THE EPIDEMIC THRESHOLD FOR VARYING TRANSPORT-INFECTION RATE

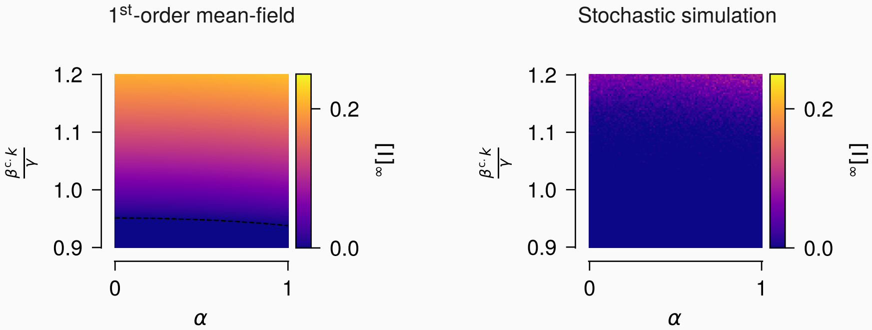
Overall, the transport process effectively amounts to $\frac{\beta^{\text{t.}}}{\beta^{\text{c.}}} \, ||p_\infty||^2 \, |\mathcal{N}|$ additional contacts to the average individual and lowers the epidemic threshold.



NON-LOCAL, FRACTIONAL TRANSPORT DYNAMICS

From a statistical point of view, human mobility patterns are characterised by heavy-tailed distributed jump-lengths, leading to non-local dynamics ($\rightarrow$ Lévy flights).

In this case we consider transport dynamics governed by a **fractional Laplacian** with exponent α.



TOWARDS A MORE REALISTIC TRANSPORT PROCESS: NON-MARKOVIAN DYNAMICS

Unlike trajectories of a random walk, those underlying human mobility tend to be **inertial**. Yet, such dynamics are inherently non-Markovian.

However: Given a random walk $(X_n)_n$ on some network $\mathcal{G}$ with k**-step memory**, there exists a Markovian random walk $(\hat{X}_n)_n$ on a higher-order network structure $\hat{\mathcal{G}}$ together with a projection Π such that the process on the original network and the one on the higher-order network under the projection $((\Pi(\hat{X}_n))_n)$ have the same one-dimensional distributions.

In the mean-field description:

$$\begin{cases} \partial_t\, p_t = -\mu\Delta^\top p_t \\ \partial_t\, [h \overset{t}{\sim} h']_t \approx \partial_t\, \|p_t\|^2\, [h]_t[h']_t \end{cases} \longrightarrow \begin{cases} \partial_t\, \hat{p}_t = -\mu\hat{\Delta}^\top \hat{p}_t \\ \partial_t\, [h \overset{t}{\sim} h']_t \approx \partial_t\, \|[\Pi]\hat{p}_t\|^2\, [h]_t[h']_t \end{cases}$$

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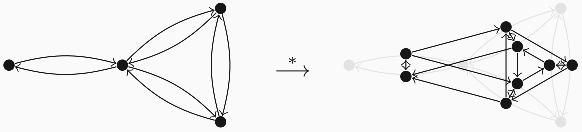
For walks with one-step-memory, an alternative (and ultimately equivalent) construction involves passing to the adjoint network:



$\mathcal{G}$ $\quad\quad$ $\hat{\mathcal{G}} = \mathcal{G}^*$

→ Markov-kernel:
$\hat{\kappa}(\,(\xi,\xi')\,|\,(x,x')) = \kappa(\,\xi'\,|\,x',x)\,\delta_{\xi,x'}$

→ projection: $\Pi : (x,x') \mapsto x'$

In such a setup, we have inertial dynamics if $\kappa(\,x\,|\,x',x) < \kappa(\,x''\,|\,x',x)$ for every $x'' \neq x$.

CONCLUSIONS

- → We have constructed an epidemic network model where the presence of a transport process gives rise to a multiplex network structure.
- → We have derived a mean-field description up to second order and from the deduced how the transport influences the epidemic threshold, under local as well as non-local (fractional) dynamics.
- → We have shown how we can incorporate more realistic non-Markovian mobility models into the mean-field description.

C. Kuehn and J. Mölter (2022). "The influence of a transport process on the epidemic threshold". **J. Math. Biol.** accepted. arXiv: `2112.04951 [nlin.AO]`

Thank you!

Deutsche Forschungsgemeinschaft

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